$\sqrt{(a+1)^2-1}$. Repeat until the circle is just tangent to the given z curve. The point of tangency gives the impedance Z_m for maximum transfer, while

$$\frac{P}{P_0}=\frac{2}{a+2}$$

as shown in Fig. 3.

Note that maximum power transfer does not occur at the point of closest approach to $Z_s{}^*$.

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A Comment on the Scattering Matrix of Cascaded 2*n*-Ports*

Epprecht[1] calculated the scattering matrix of two cascaded two-ports. Redheffer[2] does the same for the $2n$-port using nonstandard notation. This note will comment on the physical interpretation of the constituents of the resultant-scattering matrix. To use the notation of Fig. 1, the scattering matrix constituents are

$$S_{11} = S_{11}' + S_{12}'S_{11}''(1 - S_{22}'S_{11}'')^{-1}S_{21}$$
$$S_{12} = S_{12}'(1 - S_{11}'S_{22}')^{-1}S_{12}''$$
$$S_{21} = S_{21}''(1 - S_{22}'S_{11}'')^{-1}S_{21}'$$
$$S_{22} = S_{22}'' + S_{21}''S_{22}'(1 - S_{11}''S_{22}')^{-1}S_{12}''. \quad (1)$$

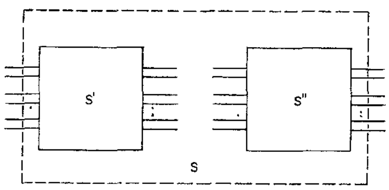

Fig. 1—S' and S'' are $n\times n$ scattering matrices of the respective networks. S is the scattering matrix of the resultant network:

$$S' = \left[\begin{array}{c|c} S_{11}' & S_{12}' \\ \hline S_{21}' & S_{22}' \end{array}\right] \qquad S'' = \left[\begin{array}{c|c} S_{11}'' & S_{12}'' \\ \hline S_{21}'' & S_{22}'' \end{array}\right].$$

The interpretation given to these formulas is that S_{11} is the bilinear transformation of S_{11} through the single primed network, and S_{22} is the bilinear transformation of S_{22}'' through the double primed network. Both of these results also follow from the definition of scattering matrix terms on the basis of matched termination (*i.e.*, if the output has a matched load, $S=0$, the input coefficient of the double primed network is S_{11}'', which is the output coefficient of the primed network). S_{12} and S_{21} are similarly interpretable, with the special case of bilaterally matched networks being the "star" multiplication of Altschuler and Kahn.[3]

It should also be noted that formulas (1) are valid when an n-port and an m-port are cascaded[4] (or interconnected).

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[1] G. W. Epprecht, "Allgemeine Aktive, Passive und Nichtreziproke Vierpole," *Tech. Mitt. PTT*, NR 5, pp. 169–173; 1959.

[2] R. M. Redheffer, "Inequalities for a matrix Riccati equation;" *J. Math. and Mech.*, vol. 3, pp. 349–367; May, 1959.

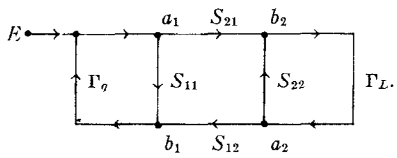
[3] H. M. Altschuler, and W. K. Kahn, "Nonreciprocal two-ports represented by modified Wheeler networks," IRE TRANS. ON MICROWAVE THEORY AND TECHNIQUES, vol. MTT-4, pp. 228–233; October, 1956.

[4] L. J. Kaplan and D. J. R. Stock, "A generalization of the matrix Riccati equation and the 'Star' multiplication of Redheffer," *J. Math. and Mech.*, vol. 6; November, 1962.

Use of Flow Graphs to Evaluate Mistermination Errors in Loss and Phase Measurements*

The purpose of this note is to show how the signal flow graph technique illustrated by Hunton[1] leads quite naturally to an expression for error due to mistermination when measuring insertion loss and phase.

We start with the flow graph used by Hunton to represent the tandem connection of generator, network and load:



With the aid of the nontouching loop rule to solve the graph, Hunton easily obtained the result

$$\frac{b_2}{E}=\frac{S_{21}}{1-\Gamma_gS_{11}-\Gamma_LS_{22}+\Gamma_g\Gamma_L(S_{11}S_{22}-S_{12}S_{21})}.$$

Very little extra work is needed to compute insertion loss and phase measurement errors due to mistermination, once the above equation is available.

E is the wave amplitude at the output port of the generator when terminated in a matched load Z_0. If V_g znd Z_g represent the Thevenin generator voltage and impedance, then

$$E=\frac{Z_0}{Z_0+Z_g}V_g.$$

Since

$$Z_g=Z_0\frac{1+\Gamma_g}{1-\Gamma_g},$$
$$E=\frac{1-\Gamma_g}{2}V_g.$$

The above equations together give

$$b_2=\frac{V_g}{2}\,\frac{S_{21}(1-\Gamma_g)}{1-\Gamma_gS_{11}-\Gamma_LS_{22}+\Gamma_g\Gamma_L(S_{11}S_{22}-S_{12}S_{21})}.$$

From the flow graph we see that $a_2=b_2\Gamma_L$. The total wave amplitude across the load is therefore

$$V_0=a_2+b_2=\frac{V_g}{2}\cdot\frac{S_{21}(1-\Gamma_g)(1+\Gamma_L)}{1-\Gamma_gS_{11}-\Gamma_LS_{22}+\Gamma_g\Gamma_L(S_{11}S_{22}-S_{12}S_{21})}.$$

Now the measured insertion ratio R_m is obtained by dividing the load voltage with network removed by the load voltage with network inserted. To remove the network, we set S_{11}, S_{22} equal to zero and S_{12}, S_{21} to unity. The result is

$$R_m=\frac{1-S_{22}\Gamma_L-S_{11}\Gamma_g+\Gamma_g\Gamma_L(S_{11}S_{22}-S_{12}S_{21})}{(1-\Gamma_g\Gamma_L)S_{21}}.$$

If the source and load were reflectionless ($\Gamma_g=\Gamma_L=0$), the corresponding insertion ratio R_0 would be just

$$R_0=\frac{1}{S_{21}}.$$

Hence, the quotient

$$Q=\frac{R_m}{R_0}=\frac{1-S_{22}\Gamma_L-S_{11}\Gamma_g+\Gamma_g\Gamma_L(S_{11}S_{22}-S_{12}S_{21})}{1-\Gamma_g\Gamma_L}$$

provides the measurement error due to network mistermination. In the common case where Γ_g and Γ_L are $\ll 1$, Q simplifies to

$$Q\sim 1+\Delta$$

where Δ, the fractional error in nepers and radians, is given by

$$\Delta=-S_{11}\Gamma_g-S_{22}\Gamma_L+\Gamma_g\Gamma_L(1+S_{11}S_{22}-S_{12}S_{21}).$$

For reciprocal structures, S_{12} is equal to S_{21}; these in turn are equal to the reciprocal of the design insertion ratio R_0.

As an example of the application of the expression for Δ, consider the measurement of a network having $|S_{11}|=|S_{22}|=0.3$ (corresponding to a VSWR of 1.85) and $|S_{12}|=|S_{21}|=1$. Then, if source and load were such that $|\Gamma_g|=|\Gamma_L|=0.02$ (VSWR of 1.04), we could expect maximum errors of 0.11 db or 0.73 degrees, depending on the phases of the S's and Γ's.

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* Received by the PGMTT, June 27, 1961

[1] J. K. Hunton, "Analysis of microwave measurement techniques by means of signal flow graphs," IRE TRANS. ON MICROWAVE THEORY AND TECHNIQUES, vol. MTT-8, pp. 206–212; March, 1960.